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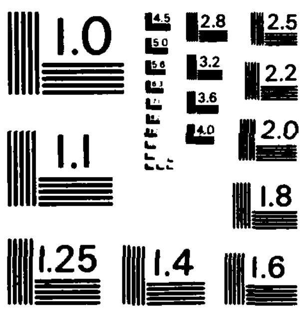

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Jack K. Hale and Arnaldo S. do Nascimento

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ORBIT CONNECTIONS IN A PARABOLIC EQUATION

by

Jack K. Hale and Arnaldo S. do Nascimento

Abstract

For all solutions of a particular scalar parabolic equation in one bounded space dimension, ~~we~~ the author discuss the global dynamics on the maximal compact invariant set and especially the orbits connecting equilibrium points.

1. Introduction. In this paper, we study some aspects of the global dynamics of the scalar parabolic equation

$$
\begin{aligned}
u_t &= u_{xx} + \lambda f(u), \quad 0 < x < \pi \\
u &= 0 \quad \text{at} \quad x = 0, \pi
\end{aligned}
\tag{1.1}
$$

where $\lambda > 0$ and f is a C^2-function.

It is shown in [6] that the initial value problem for (1.1) defines a C_0-semigroup $T_\lambda(t), t \geq 0$, on $\underline{X} = W_0^{1,2}(0,\pi)$ with $T_\lambda(t)\varphi = u(t,\cdot,\varphi)$ and $u(t,x,\varphi)$ the solution of (1.1) with $u(0,x,\varphi) = \varphi(x)$. Also, for each φ, $\{T_\lambda(t)\varphi, t \geq 0\}$ belongs to a compact set and has its ω-limit set in the set E_λ of equilibrium points

$$E_\lambda = \{\varphi \in \underline{X} : \varphi_{xx} + f(\varphi) = 0\} \tag{1.2}$$

Regularity theory for elliptic equations implies that $\varphi \in E_\lambda$ and $\varphi \in W_0^{1,2}(0,\pi) \cap W^{2,p}(0,\pi)$.

Let

$A_\lambda = \{\varphi \in \underline{X} : T_\lambda(t)\varphi$ exists and is bounded for $t \in (-\infty,\infty)\}$; that is, the set of globally defined bounded solutions of (1.1). If E_λ is a compact set, then every orbit eventually enters a neighborhood of E_λ. Classical results on dissipative processes (see, for example, Hale [3]) imply that A_λ is a maximal compact invariant set (sometimes called the attractor) for $T_\lambda(t)$, A_λ is uniformly asymptotically stable and, for any bounded set V in $\underline{X}$, the ω-limit set of V belongs to A_λ.

This remark implies that the qualitative properties of the flow defined by (1.1) is completely determined by the behavior of the flow when restricted to the attractor A_λ. Restricting the discussion of infinite dimensional dynamical systems to the attractor has been advocated for some time in connection with functional differential equations and some types of partial differential equations (see Hale [4] for a discussion and references).

Any attempt to study the flow defined by (1.1) restricted to A_λ is firstly hampered by the difficulties of understanding the bifurcation diagram for the equilibrium solutions as a function of λ in $[0,\infty)$. After one obtains the bifurcation diagram, the next step in analyzing the flow on A_λ is to determine the α-and ω-limit sets of orbits on A_λ; that is, which equilibrium points are connected by orbits.

Very few complete descriptions of the flow on A_λ have been given in the literature. Much attention has been devoted to bifurcation diagrams and local stability, but very little discussion has been devoted to the connections between orbits. The purpose of this paper is to concentrate on these connections for a particular class of f and domains Ω.

An equilibrium point φ is hyperbolic if the linear variational equation for φ does not have zero as an eigenvalue. The unstable manifold $W^u(\varphi)$ of φ has dimension k if this variational equation has k positive eigenvalues. If all equilibrium points are hyperbolic, then one can show that

$$A_\lambda = \cup_{\varphi \in E_\lambda} W^u(\varphi)$$

The problem of determining how the equilibrium points are connected on A_λ reduces therefore to a discussion of the ω-limit set of $W^u(\varphi)\setminus\{\varphi\}$.

Without further detailed hypotheses on f, the authors are aware of only one general result on the flow on A_λ. Suppose $T_\lambda(t)$ leaves the positive cone K in $\underline{X}$ invariant and one restricts the discussion to K. If $E_\lambda \cap K = \{\varphi_1, \ldots, \varphi_r\}$ with each φ_j hyperbolic, then one can show that r is odd, $\dim W^u(\varphi_{2j-1}) \cap K = 0$, $\dim W^u(\varphi_{2j}) \cap K = 1$, $j = 1,2,\ldots$ and, on $A_\lambda \cap K$, φ_{2j} is connected to $\varphi_{2j-1}, \varphi_{2j+1}$ for all j (for a proof, see Smoller [9] or Hale [5]).

To obtain more general results, we assume in this paper that f satisfies

$$f(0) = 0, \qquad f'(0) = 1 \tag{1.3}$$

$$\operatorname{sgn} f''(u) = -\operatorname{sgn} u \quad \text{for} \quad u \neq 0 \tag{1.4}$$

$$\limsup_{|u| \to \infty} \; f(u)/u \leq 0. \tag{1.5}$$

Our objective is to discuss the global flow on A_λ.

For f satisfying the above (1.3),(1.4),(1.5), Chafee and Infante [1] showed that there are exactly $2n+1$ equilibrium points $\varphi_\infty = 0$, $\varphi_j^\pm$, $j = 0,1,\ldots,n-1$, of (1.1) for $n^2 < \lambda < (n+1)^2$, $n = 0,1,2,\ldots$, each hyperbolic with φ_j^+, φ_j^- having j zeros in $(0,1)$, $\dim W^u(\varphi_j^\pm) = j$ for all j, $\dim W^u(\varphi_\infty) = n$.

The solutions $\varphi_j^\pm$ bifurcate from $\varphi_\infty = 0$ at $\lambda = j^2\pi^2$. This gives a complete description of the bifurcation diagram for equilibrium solutions for all $\lambda > 0$.

The main result of this paper is the following, where $\gamma(\varphi,\psi)$ denotes an orbit with α-limit set φ and ω-limit set ψ.

<u>Theorem 1.1.</u> <u>If</u> f <u>satisfies</u> (1.3),(1.4),(1.5), <u>and</u> $n^2 < \lambda < (n+1)^2$, <u>then</u> <u>there exist</u> $\gamma(\varphi_\infty, \varphi_j^\pm)$, $j=0,1,\ldots,n-1$, $\gamma(\varphi_j^\pm, \varphi_k^\pm)$ <u>if</u> $j > k$, j, $k=0,1,\ldots,n-1$.

For f odd, Henry [6] established the orbit connections in Theorem 1.1 for $0 \leq \lambda < 16$; that is, the λ interval which included the first three bifurcations. In an attempt to obtain orbit connections for all $\lambda \geq 0$, Hale [5] introduced an approach using the concept of lap number of Matano [8]. Our proof of Theorems 1.1 below is based on the concept of lap number and dimension theory.

2. Lap number. We first introduce the concept of lap number from Matano[8]. A real-valued function $u(r)$ on the interval $\bar{I} = [0,1]$ is said to be piecewise monotone if $\bar{I}$ can be divided into a finite number of nonoverlapping subintervals $I_1, I_2, \ldots, I_p$ $(\bigcup_{j=1}^{p} I_j = \bar{I})$ on each of which u is monotone. Let $\ell^+(u) = \min\{p, p \in \mathbb{N}\ \ I = \bigcup_{i=1}^{p} I_i$ and u/I_i is monotone increasing$\}$. Define $\ell^-(u)$ in a similar way using monotone decreasing. The nonnegative integers $\ell^+(u), \ell^-(u)$ are called the lap numbers of u. Roughly speaking, it measures the complexity of u. For a solution $u(x,t)$ of (1.1) and a fixed t its lap numbers $\ell^{\pm}(u(\cdot,t))$ are well defined and one can state the following result.

Theorem 2.1 If $f(0) = 0$ and $u(\cdot,t)$ is either a solution of (1.1) on $[0,T]$ then $\ell^+(u(\cdot,t_1)) \geq \ell^+(u(\cdot,t_2))$, $\ell^-(u(\cdot,t_1)) \geq \ell^-(u(\cdot,t_2))$ for any $0 \leq t_1 \leq t_2 < T$.

Proof of Theorem 1.1. We first derive some elementary consequences of lap number which have widespread interest. Let A_λ be either the maximal compact invariant set for Eq. (1.1) Let $M_i \in A_\lambda$, $i = 1,2,\ldots,p$, be compact invariant sets in A_λ. Following Conley [2], we say $\{M_i\}$ is a Morse decomposition of A_λ if $\varphi \in A_\lambda \setminus \bigcup_{i=1}^{p} M_i$ implies there are integers $j > k$ such that the

ω-limit set of φ is in M_k and the α-limit set of φ is in M_j.

Following Hale [7], for $\lambda_n < \lambda < \lambda_{n+1}$, let

$A_\lambda^j = \{\varphi \in A_\lambda : \{T_\lambda(t)\varphi,\ t \in \mathbb{R}$ as well as its α- and ω-limit sets have exactly j zeros in $(0,\pi)\}$.

Each set A_λ^j is compact and invariant. If we define $A_\lambda^n = \{\varphi_\infty = 0\}$, then the following result was proved in [5]. We give the proof here for completeness.

<u>Theorem 3.1.</u> <u>If</u> $\lambda_n < \lambda < \lambda_{n+1}$, <u>the sets</u> A_λ^j, $j = 0,1,\ldots,n$ <u>form a Morse decomposition of</u> A_λ. <u>Also</u>, $A_\lambda^j = \{\varphi_j^+, \varphi_j^-\}$ <u>for all</u> $j = 0,1,\ldots,n-1$.

<u>Proof</u>. Let us first prove that φ_∞ is unstable in A_λ if $\lambda_n < \lambda < \lambda_{n+1}$ for any n. Any function in the stable manifold $W^s(\alpha_\infty)$ of φ_∞ except zero has at least n zeros in $(0,1)$. We recall that A_λ is the union of the unstable manifolds of the equilibrium points. Near any equilibrium point, elements on the unstable manifolds must have no more than n zeros. Thus, Theorem 2.1 implies φ_∞ is unstable.

Now suppose that $\varphi \in A_\lambda \setminus \bigcup_{p=0}^{n} A_\lambda^p$ and $\alpha(\varphi) = \psi \in A_\lambda^j$, $\omega(\varphi) = \eta \in A_\lambda^k$. Since as $t \to -\infty$, $T_\lambda(t)\varphi \to \psi \in A_\lambda^j$, an equilibrium point with j zeros in $(0,\pi)$ and since the zeros of ψ are simple, it follows that $T_\lambda(t)\varphi$ has j zeros in $(0,\pi)$ for $t \leq -\tau$ with τ sufficiently large. Also, $T_\lambda(t)\varphi$ has $j+1$ extreme values in $(0,\pi)$. Also, $T_\lambda(t)\varphi \to \eta \in A_\lambda^k$ as $t \to \infty$, with k simple zeros and $k+1$ extreme values in $(0,\pi)$. Thus, if $\bar{\varphi} = T_\lambda(-\tau)\varphi$, then $T_\lambda(t)\bar{\varphi} \to \eta$ and Theorem 2.1 implies that the number of extreme values of $\bar{\varphi}$ must be $\geq$ the number of extreme values of η. This implies $j \geq k$. If of propagation in a homogeneous medium, $\ell=0$, may be considered as a particular case. However, for $\ell>1$ the similarity solution explodes and thus it cannot be used to represent in any sense the general Cauchy problem. This in turn signals that whenever $\ell>1$, or, as we shall find, more generally--whenever the condition (2) holds, thermal diffusion will be different. The analysis of this case is carried out in the present work.

Simply stated, the main result ensures the isothermalization of the medium to a positive average temperature.

Such a result would be natural in a finite domain wi homogeneous Neumann condition. Here it is derived for a C chy problem. It is of course drastically different from diff in a homogeneous medium or any infinite inhomogeneous mass medium, where the average temperature *is zero*. *In a standard* diffusion problem the first non-vanishing term describes the decay to "average" zero of the thermal pulse. On the other hand, in our case the calculation of how this average is approached constitutes the second term in an appropriate asymptotic expansion. We plan to report on this in the near future.

One should, however, distinguish between the approach to the average temperature $\bar{u}$ at a given point and the behavior at infinity. Whether the isothermalization of the whole space takes a finite or infinite time still remains to be answered. Note that the possibility that arbitrarily far particles have a finite temperature is physically plausible. Because there asymptotically stable in $B_\lambda^{k,j}$, $\varphi_k^{\pm}$ are unstable in $B_\lambda^{k,j}$ for $j = 0,1,\ldots,n-1$, $k = 0,1,\ldots,n$. Furthermore, $W^s(\varphi_j^{\pm}) \cap W^u(\psi)$ is open in $B_\lambda^{k,j}$ for any equilibrium point $\psi \in B_\lambda^{k,j}$.

Proof: If $d > 1$, $p > d$, the Sobolev embedding theorem implies $W_o^{1,p}(B)$ is continuously embedded in $C(B)$. If $d = 1, p = 2$, the same is true. Thus, there is an $\varepsilon > 0$ such that $|\varphi - \varphi_j^+|_{W_0^{1,p}} < \varepsilon, |\varphi - \varphi_j^-| < \varepsilon$, implies φ has exactly j zeros in $(0,1)$ and φ' has exactly $j + 1$ zeros in $(0,1)$. Theorem 2.1 implies the derivative of $\omega(\varphi)$ must have no more than $j + 1$ zeros in $(0,1)$. Thus, $\omega(\varphi)$ has no more than j zeros in $(0,1)$. Since $\omega(\varphi) \in B_\lambda^{k,j}$, it follows that $\omega(\varphi) \in A_\lambda^j$. Thus, the set A_λ^j is a maximal compact invariant set in $B_\lambda^{k,j}$ and is an attractor for the set Γ of points in $B_\lambda^{k,j}$ with j zeros in $(0,1)$. Since this set Γ is positively invariant under $T_\lambda(t)$, it follows that A_λ^j is uniformly asymptotically stable (the detailed proof is similar to ones in [3] for dissipative systems). Since $A_\lambda^j = \{\varphi_j^+, \varphi_j^-\}$ the assertions about $\varphi_j^{\pm}$ are true. From Theorem 3.1, it is clear that φ_k^+, φ_k^- are unstable in $B_\lambda^{j,k}$. This proves the first part of the lemma.

To prove the last part of the lemma, suppose $W^s(\varphi_j^{\pm}) \cap W^u(\psi)$ is not empty and contains an element $\psi \in B_\lambda^{k,j}$, $\psi \notin A_\lambda^j$. Then $\omega(\psi) \in A_\lambda^j$. Suppose $\varepsilon > 0$ is arbitrary. Since A_λ^j is uniformly asymptotically stable, there is a $\delta > 0$ such that, if φ is in the δ neighborhood of A_λ^j, then, $T_\lambda(t)\varphi$ is in the ε-neighborhood of A_λ^j and $\omega(\varphi) \in A_\lambda^j$. There is a

$t_0 = x_0(\psi, \epsilon)$ such that $T_\lambda(t_0)\psi$ is in the δ-neighborhood of A_λ^j. Continuity with respect to initial data implies a neighborhood U of ψ such that $T_\lambda(t_0)U$ is in the δ-neighborhood of A_λ^j. This completes the proof of the lemma.

<u>Lemma 3.3.</u> <u>If</u> $\lambda \in (\lambda_n, \lambda_{n+1})$, $n \geq 1$, <u>then</u> $\gamma(\varphi_\infty, \varphi_0^\pm)$, $\gamma(\varphi_j^\pm, \varphi_0^\pm)$ <u>exist for</u> $0 < j \leq n-1$.

<u>Proof:</u> The manifold $W^u(\varphi_j^+)$ is tangent at φ_j^+ to the linear subspace of $\underline{X}$ spanned by the eigenfunctions $\varphi_{j,k}$ of the operator $\partial^2/\partial x^2 + \lambda f'(\varphi_j^+)$ corresponding to the eigenvalue $\lambda_{j,k}$, $k = 0,1,\ldots,j-1$. Furthermore, $\varphi_{j,k}$ has $k-1$ zeros in $(0,1)$. Thus, there is a $\psi \in W^u(\varphi_j^+)$ such that $T_\lambda(t)\psi > \varphi_j^+$ for $t \leq -\tau$ if $\tau > 0$ is large enough. An application of the maximum principle yields $T_\lambda(t)\psi > \varphi_j^+$ for all $t \in \mathbb{R}$. Since $T_\lambda(t)\psi$ converges to an equilibrium solution as $t \to \infty$, it follows that $T_\lambda(t)\psi \to \varphi_0^+$ as $t \to \infty$, since φ_0^+ is the only equilibrium solution $> \varphi_j^+$. The same type of argument applies to φ_j^- and φ_∞ to complete the proof of the lemma.

<u>Lemma 3.4</u> <u>If</u> $n^2 < \lambda < (n+1)^2$, $n \geq 1$, <u>and</u> φ, ψ <u>are equilibrium solutions of</u> (1.1) <u>having respectively</u> $j+k$ <u>and</u> k <u>simple zeros in</u> (0,1), <u>then</u> $\psi-\varphi$ <u>has exactly</u> k <u>simple zeros in</u> (0,1).

<u>Proof:</u> It is clear from the equilibrium bifurcation problems (see [1] that

$0 < M_{j+k} = \max_{x \in [0,1]} \{\varphi(x)\} < M_j = \max_{x \in [0,1]} \{\psi(x)\}$ and $m_{j+k} = \min_{x \in [0,1]} \{\varphi(x)\} <$

$m_j = \min_{x \in [0,1]} \{\psi(x)\} < 0$. Moreover, whenever $\varphi_x(\bar{x}) = 0$, for some $\bar{x} \in (0,1)$,

either $\varphi(\bar{x}) = M_{j+k}$ or $\varphi(\bar{x}) = m$ with obvious modifications for the other cases. Also $0 < \varphi(0) < \psi(0)$.

From the above results, by plotting ψ and φ respectively it becomes clear that their graphs intersect at least at j points.

Next suppose there are $x_1 < x_2$ such that $\psi(x_1) = M_j$ $\psi(x_2) = 0$, $\psi > 0$ on (x_1,x_2). We prove that the graphs of ψ and φ cannot intersect twice on $[x_1,x_2]$. For this purpose, suppose there are y_1,y_2 such that $x_1 < y_1 < y_2 \leq x_2$ and $\psi(y_1) = \varphi(y_1)$, $\psi(y_2) = \varphi(x_2)$, $\psi(x) < \varphi(x)$, $y_1 < x < y_2$.

Let $\xi = \varphi/\psi$ on $[y_1,y_2]$. Then ξ satisfies

$$\xi_{xx} + (2\psi_x/\psi)\,\xi + \lambda\left(\frac{f(\varphi)}{\varphi} - \frac{f(\psi)}{\psi}\right)\xi = 0 \quad , \quad y_1 < x < y_2$$

$$\xi_x(y_1) > 0 \ , \ \xi_x(y_2) < 0 \quad , \quad \xi(y_1) = \xi(y_2) = 1 \ .$$

By virtue of (1.3), (1.4) and the fact that $0 < \psi < \varphi$ on (y_1,y_2), the coefficient of ξ in the above equation is negative. A maximum principle yields $\xi = 1$, which is a contradiction. The cases $y_2 = x_2 = 1$ or $x_1 = y_1 = 0$ and other similar situations can be ruled out using the same technique, thus ensuring that the graphs of ψ and φ intersect at exactly j points. This proves the lemma.

4. Proof of Theorem 1.1. In this section, we complete the proof of Theorem 1.1. To do so, we need some concepts from dimension theory.

A subset D of a topological space $\overline{X}$ is said to separate $\overline{X}$ if $\overline{X} \setminus D$ is disconnected. The proof the following lemma may be found in [7].

Lemma 4.1. Any n-dimensional manifold cannot be separated by a subset of dimension less than or equal to n-2.

The following result is the first step in the proof of Theorem 1.1.

Lemma 4.2. For $\lambda_n < \lambda < \lambda_{n+1}$, $n \geq 1$, there exist orbits

$$\gamma(\varphi_\infty, \varphi_j^+),\ \gamma(\varphi_\infty, \varphi_j^-),\ j = 0,1,\ldots,n-1.$$

Proof: For this proof, let $\overline{X} = W_0^{1,p}(\Omega)$. Existence of the orbits $\gamma(\varphi_\infty, \varphi_0^\pm)$ has been proved in Lemma 3.3. From the definition of $B_\lambda^{j,k}$, we have $B_\lambda^{n,0} = A_\lambda$,

$$B_\lambda^{n,j+1} = B_\lambda^{n,j} - [W^s(\varphi_j^+) \cup W^s(\varphi_j^-)]$$

and φ_j^+, φ_j^- are uniformly asymptotically stable in $B_\lambda^{n,j}$ from Lemma 3.2. Furthermore Lemma 3.2 implies the sets $\mathcal{O}_{\infty,\pm}^j = W^s(\varphi_j^\pm) \cap W^u(\varphi_\infty)$ are disjoint open sets in $B_\lambda^{n,j} \cap W^u(\varphi_\infty)$.

Since $W^u(\varphi_\infty) \subseteq B_\lambda^{n,0}$, the sets $\mathcal{O}_{\infty,\pm}^0$ are disjoint open sets in the connected n-dimensional manifold $W^u(\varphi_\infty)$. It follows that $W_0^u(\varphi_\infty) \overset{\text{def}}{=} B_\lambda^{n,1} \cap W^u(\varphi_\infty)$ separates $W^u(\varphi_\infty)$ and Lemma 4.1 implies that

$$n \geq \dim W_0^u(\varphi_\infty) \geq n-1.$$

We need a local representation of $W^u(\varphi_\infty)$. From stable and unstable manifold theory, if $\varphi_{\infty,j}$, $j=0,1,\ldots,n-1$ are eigenfunctions corresponding to the eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$ and

$$\varphi(\alpha) = \sum_{j=0}^{n-1} \alpha_j \varphi_{\infty,j} \quad , \quad \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{R}^n$$

$$\Sigma = \{\psi \in \underline{X} : \psi = \varphi(\alpha),\ \alpha \in \mathbb{R}^n\}$$

then there is a Lipschitz continuous function $h : \Sigma \to W^u(\varphi_\infty)$, $|h(\varphi)|_{\underline{X}} = o(|\varphi|_{\underline{X}})$ as $|\varphi|_{\underline{X}} \to 0$ and an $\varepsilon_0 > 0$ such that the set

$$V(\varphi_\infty) = \{\psi : \psi = \varphi(\alpha) + h(\varphi(\alpha)),\ |\alpha| < \varepsilon_0\}$$

is a neighborhood of φ^∞ in $W^u(\varphi_\infty)$ where $\alpha = \max|\alpha_i|$.

Now suppose that no point in $W_0^u(\varphi_\infty) \cap V(\varphi_\infty)$ has ω-limit set in $A_\lambda^1 = \{\varphi_1^+, \varphi_1^-\}$. Choose $\alpha_2 = \ldots = \alpha_{n-1} = 0$ and choose real numbers $\varepsilon_0 > 0$, $\varepsilon_1 > 0$ so small that, for every $0 < |\alpha_0| < \varepsilon_0$, $0 < |\alpha_1| < \varepsilon_1$, the function $\eta(\alpha) = \varphi_0(\alpha) + h(\varphi_0(\alpha))$, $\varphi_0(\alpha) = \alpha_0\varphi_{\infty,0} + \alpha_1\varphi_{\infty,1}$ has no more than one zero in $(0,1)$. This can be done since $h(\varphi) = o(|\varphi|_{\underline{X}})$ as $|\varphi|_{\underline{X}} \to 0$.

Let U be the set $\{\eta(\alpha),\ 0 \le |\alpha_0| \le \varepsilon_0,\ 0 \le |\alpha_1| \le \varepsilon_1\}$. Since no point in $W_0^u(\varphi_\infty) \cap \mathscr{V}(\varphi_\infty)$ has ω-limit set in A_λ^1, Theorem 2.1 implies that $\omega(\psi) \subseteq A_\lambda^0$ for $\psi \in U \setminus \{\varphi_\infty\}$. Thus, in a neighborhood of $\varphi_\infty, W^u(\varphi_\infty)$ is not separated by $W_0^u(\varphi_\infty)$ since $W_0^u(\varphi_\infty)$ is determined by only the n-2 numbers $\alpha_2, \ldots, \alpha_{n-1}$. This is a contradiction and shows that there is an element in $W_0^u(\varphi_\infty) \cap \mathscr{V}(\varphi_\infty)$ with ω-limit set in A_λ^1. Since $A_\lambda^1 = \{\varphi_1^+, \varphi_1^-\}$, this implies either $\gamma(\varphi_\infty,\varphi_1^+)$ or $\gamma(\varphi_\infty,\varphi_1^-)$ exists. Suppose $\gamma(\varphi_\infty,\varphi_1^+)$ exists. Then there is a solution $u(t,x)$ of the equation with $u(t,x) \to \varphi_\infty(x)$ as $t \to -\infty$, $u(t,x) \to \varphi_1^+(x)$ as $t \to -\infty$. But then $u(t,\pi-x) \to \varphi_1^-(x) = \varphi_1^+(\pi-x)$ as $t \to \infty$ and $\gamma(\varphi_\infty,\varphi_1^-)$ exists. The same argument applies if one assumes $\gamma(\varphi_\infty,\varphi_1^-)$ exists.

From Lemma 3.2, φ_1^+,φ_1^- are uniformly asymptotically stable in $B_\lambda^{n,1}$ and $\mathcal{O}^1_{\infty,\pm}$ are disjoint open sets in $W_0^u(\varphi_\infty) = B_\lambda^{n,1} \cap W^u(\varphi_\infty)$. Since $n-1 \le \dim W_0^u(\varphi_\infty) \le n$, it follows that $n-1 \le \dim \mathcal{O}^1_{\infty,\pm} \le n$. Lemma 3.2 implies φ_∞ is unstable in $W_0^u(\varphi_\infty)$. Choose the connected component $C_\lambda^{n,1}$ of $B_\lambda^{n,1}$ that contains both φ_1^+ and φ_1^-, define $\tilde{B}_\lambda^{n,2} = B_\lambda^{n,2} \cap C_\lambda^{n,1}$, $W_1^u(\varphi_\infty) = \tilde{B}_\lambda^{n,2} \cap W^u(\varphi_\infty)$. Then $W_1^u(\varphi_\infty)$ separates $W_0^u(\varphi_\infty)$ and

$n-2 \le \dim W_1^u(\varphi_\infty) \le n-1$. Exactly as before, one shows that there is an orbit in $W_1^u(\varphi_\infty)$ connecting φ_∞ to A_λ^2. Then one argues as above to see that $\gamma(\varphi_\infty,\varphi_2^+)$, $\gamma(\varphi_\infty,\varphi_1^+)$ exist.

This process is repeated to obtain a proof of Lemma 4.2.

To complete the proof of Theorem 1.1, we must show $\gamma(\varphi_j^\pm,\varphi_k^\pm)$ exists if $j > k$. To prove this, fix j and consider the compact invariant set $B_\lambda^{j,0}$. We know from Lemma 3.2 that α_0^+,α_0^- are uniformly asymptotically stable in $B_\lambda^{j,0}$ and φ_j^+,φ_j^- are unstable with $\dim W^u(\varphi_j^\pm) = j$. If $u = \varphi_j^+ + v$, then $v = 0$ has unstable manifold of dimension j and Lemma 3.4 implies that the equilibrium points $\varphi = \varphi_k^\pm - \varphi_j^+$ for $0 \le k < j$ have

k zeros in $(0,1)$. One can now repeat the same type of argument as above to obtain the desired connections taking first $W^s(\varphi_0^{\pm} - \varphi_j^{+})$ away from $B_\lambda^{0,j}$, then $W^s(\varphi_1^{\pm} - \varphi_j^{+})$, etc.

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